MILP sensitivity analysis for the cost coefficients

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MILP sensitivity analysis

Consider the MILP

$$\max\{\nu(x,y) \mid (x,y) \in \mathcal{X}\}, \tag{Π}$$

with $\nu(x,y) = cx + hy$ and $\mathcal{X} = \{(x,y) \in \mathbb{Z}^n \times \mathbb{R}^p \mid Ax + Gy \leq b, x \geq 0, y \geq 0\}$.
Let (x^*, y^*) be an optimal solution.

For $K \subseteq \{1, 2, \cdots, n\}$, the parameterized MILP replaces the objective function coefficients $c_k, k \in K$ by $c_k + \Delta_k, k \in K$

$$\max\{\nu_\Delta(x,y) \mid (x,y) \in \mathcal{X}\}, \tag{Π_Δ}$$

with $\Delta = (\Delta_k : k \in K)$ and $\nu_\Delta(x,y) = cx + hy + \sum_{k \in K} \Delta_k x_k$.

Definition 1: The sensitivity region

The sensitivity region is the region of changes to the cost coefficients such that (x^*, y^*) remains optimal to (Π), i.e.

$$\Omega(\mathcal{X}) = \left\{\Delta \in \mathbb{R}^{|K|} \mid \nu_\Delta(x,y) \leq \nu_\Delta(x^*, y^*), (x,y) \in \mathcal{X}\right\}.$$

MILP sensitivity analysis

For LP, sensitivity analysis only requires re-computing reduced costs and re-optimizing.

The MILP is equivalent to the LP

$$\max\{\nu(x,y) \mid (x,y) \in \text{conv}(\mathcal{X})\}, \tag{Π}$$

However, conv($\mathcal{X}$) is generally unknown and

$$\left\{\Delta \in \mathbb{R}^{|K|} \mid \nu_\Delta(x,y) \leq \nu_\Delta(x^*,y^*), (x,y) \in \text{conv}(\mathcal{X})\right\}$$
$$\neq \left\{\Delta \in \mathbb{R}^{|K|} \mid \nu_\Delta(x,y) \leq \nu_\Delta(x^*,y^*), (x,y) \in \mathcal{X}^{LP}\right\},$$

where $\mathcal{X}^{LP} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^p \mid Ax + Gy \leq b, x \geq 0, y \geq 0\}$.

Applications

- Electricity scheduling and unit commitment: Minimize costs, minimize environmental damage (emissions), maximize reliability (minimize unmet demand and/or black-outs).
- Vehicle routing: Minimize fuel consumption, maximize customer satisfaction (minimize lost sales and/or delays).
- Design of housing mobility programs: Maximize economic benefits, minimize fairness disparities.

Costs are hard to quantify/uncertain/hard to predict/vary over time.

Capital budgeting

Procure R&D contracts by optimally allocating limited funds to suggested projects (Petersen (1967)).

The problem can be formulated as a binary ILP:

$$\max \left\{ \sum_{j=1}^{n} c_j x_j \,\middle|\, \sum_{i=1}^{m} a_{ij} x_j \leq b_i, i = 1, \ldots, m, x_j \in \{0,1\}, j = 1, \ldots, n \right\},$$

where $x_j = 1$ if project j is selected and $x_j = 0$ otherwise. Moreover, c_j is the value of the contract, a_{ij} is a cost and b_i is the budget.

Future contract values are not known with certainty but are only estimated.

Production lot sizing

Determine an optimal production plan for various types of items over a finite planning horizon.

The problem can be formulated as a MILP:

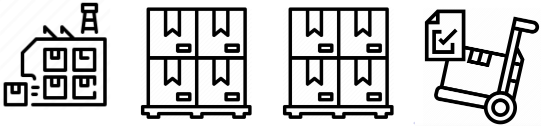
Minimize

- linear inventory and backlog costs
- fixed setup costs
- piecewise linear and convex transportation costs

subject to

- demand coverage
- capacity constraints on the time spent on production and setup

The problem may have to be solved repeatedly over time for varying costs.



Changes to a single coefficient

For $k \in \{1, 2, \cdots, n\}$, the parameterized MILP replaces the objective function coefficient c_k by $c_k + \Delta$

$$\max\{\nu_\Delta(x, y) \mid (x, y) \in \mathcal{X}\}, \qquad (\Pi_\Delta)$$

with $\Delta \in \mathbb{R}$ and $\nu_\Delta(x, y) = cx + hy + \Delta x_k$.

Definition 2: The sensitivity interval

The sensitivity interval is

$$[lb, ub] = \{\Delta \in \mathbb{R} \mid \nu_\Delta(x, y) \le \nu_\Delta(x^*, y^*), (x, y) \in \mathcal{X}\}.$$

For $\hat{\mathcal{X}} \subseteq \mathcal{X}$, define

$$lb(\hat{\mathcal{X}}) = \sup\left\{\frac{\nu(x^*, y^*) - \nu(x, y)}{x_k - x_k^*} \mid (x, y) \in \hat{\mathcal{X}}, x_k < x_k^*\right\} \le 0,$$

$$ub(\hat{\mathcal{X}}) = \inf\left\{\frac{\nu(x^*, y^*) - \nu(x, y)}{x_k - x_k^*} \mid (x, y) \in \hat{\mathcal{X}}, x_k > x_k^*\right\} \ge 0.$$

Then, $[lb, ub] = [lb(\mathcal{X}), ub(\mathcal{X})] \subseteq [lb(\hat{\mathcal{X}}), ub(\hat{\mathcal{X}})]$.

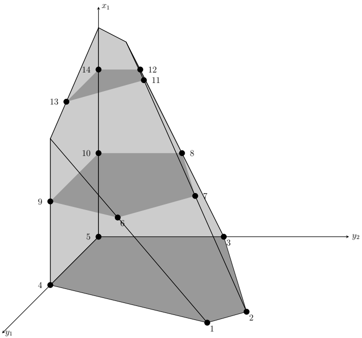
Determine $\hat{\mathcal{X}} \subset \mathcal{X}$ with $|\hat{\mathcal{X}}| \ll |\mathcal{X}|$ such that $[lb(\mathcal{X}), ub(\mathcal{X})] = [lb(\hat{\mathcal{X}}), ub(\hat{\mathcal{X}})]$.

Example 1

Consider the MILP

$$\max\{3x_1 + y_1 + y_2 \mid (x, y) \in \mathcal{X}\},$$

An optimal solution is $(x^*, y^*) = (x^{11}, y^{11}) = (2, 1/3, 2/3)$ with $\nu^* = \nu(x^*, y^*) = 7$.



(a) Solution space $\mathcal{X}$.

Coeff.	$lb(\hat{\mathcal{X}})$	$ub(\hat{\mathcal{X}})$	argmax	argmin
$c_1 = 3$	-1	∞	1, 2, 6, 7	none
$h_1 = 1$	-3/2	0	12	13
$h_2 = 1$	0	1	13	2, 7

(b) Sensitivity intervals.

The sensitivity regions can be determined by the set of vertices of $\text{conv}(\mathcal{X})$, i.e. for $\hat{\mathcal{X}} = \{(x^1, y^1), \ldots, (x^{14}, y^{14})\}$, $[lb, ub] = [lb(\hat{\mathcal{X}}), ub(\hat{\mathcal{X}})]$. But $\text{conv}(\mathcal{X})$ is generally unknown!

Multi-objective optimization

Consider the MO-MILP

$$\max \{z(x,y) \mid (x,y) \in \mathcal{X}\}, \qquad (\Pi_{MO})$$

with feasible *solution* $(x,y) \in \mathcal{X}$ and its corresponding *objective point* $z(x,y) = \big(z_1(x,y), \ldots z_q(x,y)\big)$, where $z_i(x,y) = c^i x + h^i y$, $i \in \{1, \ldots, q\}$.

We refer to the feasible set in *solution space* $\mathcal{X}$ and in *objective space*,

$$\mathcal{Z} = \{z(x,y) \in \mathbb{R}^q \mid (x,y) \in \mathcal{X}\}.$$

We compare feasible solutions by comparing their objective points. For objective points $z^1, z^2 \in \mathbb{R}^q$,

$$z^1 \succ z^2 \quad \text{if and only if} \quad z_i^1 \geq z_i^2,\ i = 1, \ldots, q \text{ and } z^1 \neq z^2.$$

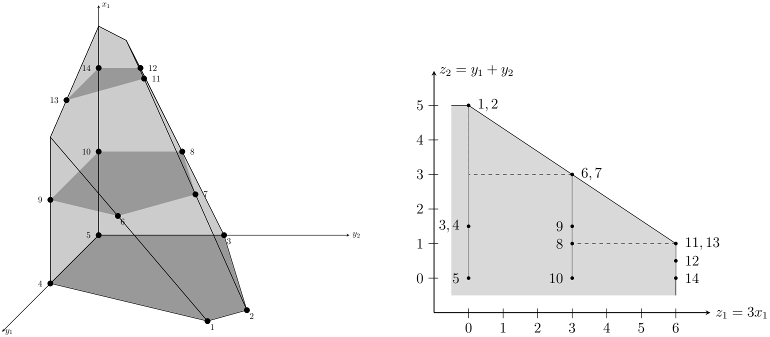
A point z^2 is said to be *dominated* by z^1 if $z^1 \succ z^2$.

We define the *efficient set* and the *nondominated set*, respectively,

$$\mathcal{X}_E = \{(x,y) \in \mathcal{X} \mid \nexists (\hat{x}, \hat{y}) \in \mathcal{X} : z(\hat{x}, \hat{y}) \succ z(x,y)\},$$
$$\mathcal{Z}_N = \{z(x,y) \in \mathcal{Z} \mid (x,y) \in \mathcal{X}_E\}$$

Multi-objective optimization

A point $z \in \mathcal{Z}_N$ is a *supported* nondominated point if it is on the boundary of the polyhedron $\mathcal{Z}^{\leqq} = \operatorname{conv}(\mathcal{Z}_N + \{z \in \mathbb{R}^q : z \leq 0\})$; otherwise it is *unsupported*.
A supported nondominated point z is *extreme* if it is a vertex of $\mathcal{Z}^{\leqq}$; otherwise it is *nonextreme*.



(a) Solution space $\mathcal{X}$. (b) Objective space.

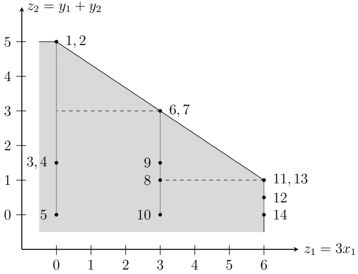
A (minimal) efficient set is $\mathcal{X}_E = \{(x^1, y^1), (x^6, y^6), (x^{11}, y^{11})\}$ and the nondominated set is $\mathcal{Z}_N = \{z^1, z^6, z^{11}\}$, where z^1 and z^{11} are extreme nondominated points and z^6 is a supported nonextreme nondominated point.

Example 1

Consider the MO-MILP

$$\max\{(3x_1, y_1 + y_2) \mid (x, y) \in \mathcal{X}\}.$$

with efficiency set $X_E^+ = \{(x^1, y^1), (x^6, y^6), (x^{11}, y^{11})\}$.



(a) Objective space.

Coeff.	$lb(\mathcal{X}_E^+)$	$ub(\mathcal{X}_E^+)$	argmax	argmin
$c_1 = 3$	-1	∞	1, 6	none
$h_1 = 1$	$-\infty$	0	none	13
$h_2 = 1$	0	1	13	2, 7

(b) Sensitivity intervals.

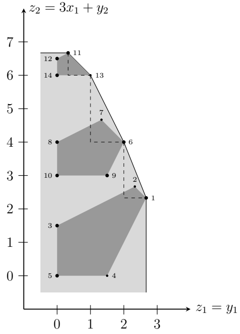
For changes to c_1, the sensitivity regions can be determined by $\mathcal{X}_E^+$, i.e. $[lb, ub] = [lb(\mathcal{X}_E^+), ub(\mathcal{X}_E^+)]$.

Example 1

Consider the MO-MILP

$$\max\{(y_1, 3x_1 + y_2) \mid (x, y) \in \mathcal{X}\}.$$

with efficiency set $X_E^+ = \{L(1,2), L(6,7), L(11,13)\}$, where $L(i,j)$ denotes the line between (x^i, y^i) and (x^j, y^j).



(a) Objective space.

Coeff.	$lb(\mathcal{X}_E^+)$	$ub(\mathcal{X}_E^+)$	argmax	argmin
$c_1 = 3$	-1	∞	1, 6	none
$h_1 = 1$	$-\infty$	0	none	13
$h_2 = 1$	0	1	13	2, 7

(b) Sensitivity intervals.

For changes to h_1, $[lb, ub] \subset [lb(\mathcal{X}_E^+), ub(\mathcal{X}_E^+)]$.

Bi-objective auxillary problems

Consider the bi-objective MILP Π^+_{MO}

$$\begin{aligned} \max\ & z^+(x,y) = (z_1(x,y), z_2(x,y)) = (c_k x_k, \sum_{\substack{j=1 \\ j\neq k}}^{n} c_j x_j + hy) \\ \text{st}\ & (x,y) \in \mathcal{X} \end{aligned} \qquad (\Pi^+_{\mathrm{MO}})$$

and denote by $\mathcal{X}^+_E$ the efficient set.

Lemma 1

If $c_k > 0$, define

$$lb^+ = \max\left(lb(\mathcal{X}^+_E), -c_k\right), \quad ub^+ = ub(\mathcal{X}^+_E).$$

Then, $lb \leq lb^+ \leq ub^+ = ub$. Moreover, if $lb^+ = lb(\mathcal{X}^+_E)$ then $lb^+ = lb$.
If $c_k < 0$, define

$$lb^+ = lb(\mathcal{X}^+_E), \quad ub^+ = \min\left(ub(\mathcal{X}^+_E), -c_k\right).$$

Then, $lb = lb^+ \leq ub^+ \leq ub$. Moreover, if $ub^+ = ub(\mathcal{X}^+_E)$ then $ub^+ = ub$.

If $c_k > 0$ and $c_k + \Delta \geq 0$, then $[lb^+, ub^+] = [lb, ub]$; otherwise $[lb^+, ub^+] \subseteq [lb, ub]$.

Bi-objective auxillary problems

Proof:

It is sufficient to show that $[lb^+, ub^+] \subseteq [lb(\mathcal{X}), ub(\mathcal{X})]$.

Conversely, now assume that $\exists \Delta : \Delta \in [lb^+, ub^+] \backslash [lb(\mathcal{X}), ub(\mathcal{X})]$ implying that $\exists (\hat{x}, \hat{y}) \in \mathcal{X} : \nu_\Delta(\hat{x}, \hat{y}) > \nu_\Delta(x^*, y^*)$.

Note that since $\Delta \in [lb^+, ub^+] \subseteq [lb(\mathcal{X}_E^+), ub(\mathcal{X}_E^+)]$, we have $\nu_\Delta(x, y) \leq \nu_\Delta(x^*, y^*), (x, y) \in \mathcal{X}_E^+$. Hence, $(\hat{x}, \hat{y}) \notin \mathcal{X}_E^+$ and there exists $(\bar{x}, \bar{y}) \in \mathcal{X}_E^+$ such that

(i) $c_k \bar{x}_k \geq c_k \hat{x}_k$ and
(ii) $\sum_{j \neq k} c_j \bar{x}_j + h\bar{y} \geq \sum_{j \neq k} c_j \hat{x}_j + h\hat{y}$.

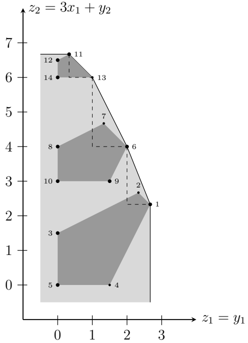
We multiply (i) by $(c_k + \Delta)/c_k \geq 0$ and add it to (ii) to obtain $\nu_\Delta(\bar{x}, \bar{y}) \geq \nu_\Delta(\hat{x}, \hat{y})$. But then $\nu_\Delta(\bar{x}, \bar{y}) > \nu_\Delta(x^*, y^*)$, contradicting $\Delta \in [lb(\mathcal{X}_E^+), ub(\mathcal{X}_E^+)]$.

Example 1

Consider the MO-MILP

$$\max\{(-y_1, 3x_1 + y_2) \mid (x, y) \in \mathcal{X}\}.$$

with efficiency set $X_E^- = \{L(2,3), L(7,8), L(11,12)\}$.



(a) Objective space.

Coeff.	$lb(\mathcal{X}_E^+ \cup \mathcal{X}_E^-)$	$ub(\mathcal{X}_E^+ \cup \mathcal{X}_E^-)$	argmax	argmin
$c_1 = 3$	-1	∞	1, 6	none
$h_1 = 1$	-3/2	0	12	13
$h_2 = 1$	0	1	13	2, 7

(b) Sensitivity intervals.

For changes to h_1, $[lb, ub] = [lb(\mathcal{X}_E^+ \cup \mathcal{X}_E^-), ub(\mathcal{X}_E^+ \cup \mathcal{X}_E^-)]$

Bi-objective auxillary problems

Consider the bi-objective problem Π^{-}_{MO}

$$\begin{aligned} \max\ & z^{-}(x,y) = (-z_1(x,y), z_2(x,y)) = (-c_k x_k, \sum_{\substack{j=1\\ j\neq k}}^{n} c_j x_j + hy) \\ \text{st}\ & (x,y) \in \mathcal{X} \end{aligned} \qquad (\Pi^{+}_{\text{MO}})$$

and denote by $\mathcal{X}^{-}_{E}$ the efficient set.

Theorem 1

For $c_k \neq 0$, the sensitivity interval $[lb, ub]$ is

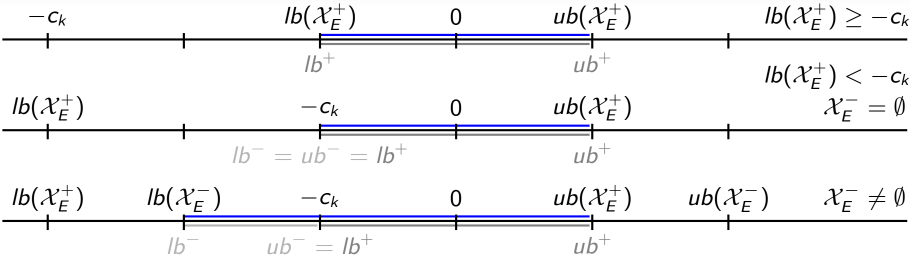
$$lb = \begin{cases} -c_k, & \text{if } c_k > 0,\ lb(\mathcal{X}^{+}_{E}) < -c_k \text{ and } \mathcal{X}^{-}_{E} = \emptyset, \\ lb(\mathcal{X}^{-}_{E}), & \text{if } c_k > 0,\ lb(\mathcal{X}^{+}_{E}) < -c_k \text{ and } \mathcal{X}^{-}_{E} \neq \emptyset, \\ lb(\mathcal{X}^{+}_{E}), & \text{otherwise,} \end{cases}$$

and

$$ub = \begin{cases} -c_k, & \text{if } c_k < 0,\ ub(\mathcal{X}^{+}_{E}) > -c_k \text{ and } \mathcal{X}^{-}_{E} = \emptyset, \\ ub(\mathcal{X}^{-}_{E}), & \text{if } c_k < 0,\ ub(\mathcal{X}^{+}_{E}) > -c_k \text{ and } \mathcal{X}^{-}_{E} \neq \emptyset, \\ ub(\mathcal{X}^{+}_{E}), & \text{otherwise.} \end{cases}$$

Bi-objective auxillary problems

$c_k > 0$



Note that

$$[lb^+, ub^+] = \{\Delta \in [lb(\mathcal{X}_E^+), ub(\mathcal{X}_E^+)] \mid \text{sgn}(c_k + \Delta) \in \{\text{sgn}(c_k), 0\}\},$$

$$[lb^-, ub^-] = \{\Delta \in [lb(\mathcal{X}_E^-), ub(\mathcal{X}_E^-)] \mid \text{sgn}(c_k + \Delta) \in \{-\text{sgn}(c_k), 0\}\},$$

and

$$[lb, ub] = [lb^+, ub^+] \cup [lb^-, ub^-].$$

From efficiency solutions to nondominated points

Denote the nondominated sets of Π^+_{MO} and Π^-_{MO}, respectively, by

$$\mathcal{Z}^+_N = \{z^+(x,y) \mid (x,y) \in \mathcal{X}^+_E\},$$
$$\mathcal{Z}^- = \{z^+(x,y) \mid (x,y) \in \mathcal{X}^-_E\},$$

For $\hat{\mathcal{Z}} \subseteq \mathcal{Z}$, we define

$$lb(\hat{\mathcal{Z}}) = \sup\left\{\frac{c_k(z_1^* + z_2^* - z_1 - z_2)}{z_1 - z_1^*} \mid (z_1, z_2) \in \hat{\mathcal{Z}}, \frac{z_1}{c_k} < \frac{z_1^*}{c_k}\right\},$$
$$ub(\hat{\mathcal{Z}}) = \inf\left\{\frac{c_k(z_1^* + z_2^* - z_1 - z_2)}{z_1 - z_1^*} \mid (z_1, z_2) \in \hat{\mathcal{Z}}, \frac{z_1}{c_k} > \frac{z_1^*}{c_k}\right\}.$$

Corollary 1

For $c_k \neq 0$,

$lb(\mathcal{X}^+_E) = lb(\mathcal{Z}^+_N),\ ub(\mathcal{X}^+_E) = ub(\mathcal{Z}^+_N),\ lb(\mathcal{X}^-_E) = lb(\mathcal{Z}^-),\ ub(\mathcal{X}^-_E) = ub(\mathcal{Z}^-).$

Extreme nondominated points

Denote by $\mathcal{Z}^+_{N,e}$ and $\mathcal{Z}^-_e$, respectively, the extreme points of the polyhdra

$$\mathcal{Z}^+_{\leq} = \text{conv}\left(\mathcal{Z}^+_N + \{z \in \mathbb{R}^q : z \leq 0\}\right),$$
$$\mathcal{Z}^-_{\leq} = \text{conv}\left(\mathcal{Z}^- + \{z \in \mathbb{R}^q : z \leq 0\}\right).$$

Corollary 2

For $c_k \neq 0$,

$$lb(\mathcal{X}^+_E) = lb(\mathcal{Z}^+_{N,e}),\ ub(\mathcal{X}^+_E) = ub(\mathcal{Z}^+_{N,e}),\ lb(\mathcal{X}^-_E) = lb(\mathcal{Z}^-_e),\ ub(\mathcal{X}^-_E) = ub(\mathcal{Z}^-_e).$$

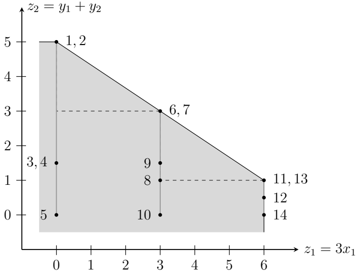
In fact, it is sufficient to consider extreme nondominated points adjacent to the optimal solution.

Example 1

Consider the MO-MILP

$$\max\{(3x_1, y_1 + y_2) \mid (x, y) \in \mathcal{X}\}.$$

with the set of extreme nondominated points $\mathcal{Z}^+_{N,e} = \{(x^1, y^1), (x^{11}, y^{11})\}$.



(a) Objective space.

Coeff.	$lb(\mathcal{Z}^+_{N,e})$	$ub(\mathcal{Z}^+_{N,e})$	argmax	argmin
$c_1 = 3$	-1	∞	1	none
$h_1 = 1$	$-\infty$	0	none	13
$h_2 = 1$	0	1	13	2

(b) Sensitivity intervals.

Changes to two coefficients

Consider the four tri-objective problems $\Pi_{\text{MO}}^{++}, \Pi_{\text{MO}}^{+-}$

$$\max \{z^{++}(x,y) = (c_k x_k, c_l x_l, \sum_{i \neq k,l} c_i x_i + hy) \mid (x,y) \in \mathcal{X}\},$$

$$\max \{z^{+-}(x,y) = (c_k x_k, -c_l x_l, \sum_{i \neq k,l} c_i x_i + hy) \mid (x,y) \in \mathcal{X}\},$$

and similarly, Π_{MO}^{-+} and Π_{MO}^{--}. Denote by $\mathcal{X}_E^{++}, \mathcal{X}_E^{+-}, \mathcal{X}_E^{-+}$ and $\mathcal{X}_E^{--}$ the efficient sets.

Define the sets

$$\Omega^{++} = \{(\Delta_k, \Delta_l) \in \Omega(\mathcal{X}_E^{++}) \mid \operatorname{sgn}(c_k + \Delta_k) \in \{\operatorname{sgn}(c_k), 0\}, \operatorname{sgn}(c_l + \Delta_l) \in \{\operatorname{sgn}(c_l), 0\}\},$$

$$\Omega^{+-} = \{(\Delta_k, \Delta_l) \in \Omega(\mathcal{X}_E^{+-}) \mid \operatorname{sgn}(c_k + \Delta_k) \in \{\operatorname{sgn}(c_k), 0\}, \operatorname{sgn}(c_l + \Delta_l) \in \{-\operatorname{sgn}(c_l), 0\}\},$$

and similarly, Ω^{-+} and Ω^{--}.

Theorem 2

For $c_k, c_l \neq 0$, the sensitivity region is $\Omega = \Omega^{++} \cup \Omega^{+-} \cup \Omega^{-+} \cup \Omega^{--}$.

Changes to K coefficients requires at most 2^K $(K+1)$-objective auxillary problems.

Capital budgeting

j	c_j	Δc_j	j	c_j	Δc_j	j	c_j	Δc_j
1	560	$[-\infty, 43]$	18	115	$[-\infty, 41]$	35	71	$[-13, +\infty]$
2	1125	$[-\infty, 189]$	19	82	$[-44, +\infty]$	36	49	$[-19, +\infty]$
3	300	$[-\infty, 62]$	20	22	$[-13, +\infty]$	37	108	$[-58, +\infty]$
4	620	$[-74, +\infty]$	21	631	$[-\infty, 72]$	38	116	$[-18, +\infty]$
5	2100	$[-\infty, 809]$	22	132	$[-\infty, 57]$	39	90	$[-64, +\infty]$
6	431	$[-140, +\infty]$	23	420	$[-76, +\infty]$	40	738	$[-382, +\infty]$
7	68	$[-\infty, 18]$	24	86	$[-\infty, 39]$	41	1811	$[-884, +\infty]$
8	328	$[-85, +\infty]$	25	42	$[-13, +\infty]$	42	430	$[-286, +\infty]$
9	47	$[-28, +\infty]$	26	103	$[-19, +\infty]$	43	3060	$[-2076, +\infty]$
10	122	$[-\infty, 33]$	27	215	$[-75, +\infty]$	44	215	$[-38, +\infty]$
11	322	$[-87, +\infty]$	28	81	$[-29, +\infty]$	45	58	$[-\infty, 13]$
12	196	$[-19, +\infty]$	29	91	$[-71, +\infty]$	46	296	$[-\infty, 77]$
13	41	$[-32, +\infty]$	30	26	$[-\infty, 13]$	47	620	$[-181, +\infty]$
14	25	$[-\infty, 14]$	31	49	$[-43, +\infty]$	48	418	$[-191, +\infty]$
15	425	$[-248, +\infty]$	32	420	$[-285, +\infty]$	49	47	$[-18, +\infty]$
16	4260	$[-1389, +\infty]$	33	316	$[-\infty, 18]$	50	81	$[-61, +\infty]$
17	416	$[-245, +\infty]$	34	72	$[-31, +\infty]$			

- $x_1^* = 0$ remains optimal for any decrease in volume
- $x_4^* = 1$ remains optimal for any increase in volume

changes	total CPU (sec)	avg CPU (sec)
1	8	0.15
2	400	0.33
3	820	0.72

Production lot sizing

Table: CPU (sec), 3 items, 8 time periods

Costs	Var	Single variable Average	Std.dev.	Median	Min	Max	All variables Average	Std.dev.	Median	Min	Max
inventory	cont	4.39	12.95	1.56	0.42	161.25	105.40	139.07	32.86	21.74	438.37
backlog	cont	2.77	12.23	1.16	0.41	187.48	66.37	77.71	26.40	16.08	250.06
setup	bin	0.92	2.26	0.31	0.13	24.27	22.13	27.96	7.84	4.81	78.25
transport	int	2.00	1.75	1.32	0.44	12.84	48.03	34.12	31.42	20.75	112.00
transport	int	1.52	1.11	0.94	0.41	5.11	36.40	25.62	22.53	14.55	86.38
All		2.32	8.16	1.03	0.13	187.48	278.06	293.11	120.26	80.73	907.83

Table: CPU (sec), 3 items, 12 time periods

Costs	Var	Single variable Average	Std.dev.	Median	Min	Max	All variables Average	Std.dev.	Median	Min	Max
inventory	cont	12.48	39.00	4.32	0.61	661.20	449.15	499.17	259.29	49.18	1417.63
backlog	cont	6.89	14.10	3.20	0.63	215.48	247.99	288.60	126.77	34.85	860.83
setup	bin	1.96	7.73	0.80	0.16	138.66	70.60	83.24	44.38	14.52	294.11
transport	int	6.28	7.58	3.28	0.63	40.69	226.20	256.53	148.37	42.37	880.45
transport	int	5.39	7.33	2.18	0.64	49.39	194.06	248.41	105.96	33.02	846.77
All		6.60	19.72	2.15	0.16	661.20	1179.58	1316.76	671.26	172.01	4272.51

MO-MILP sensitivity analysis

Consider the MO-MILP

$$\max\{\nu(x,y) \mid (x,y) \in \mathcal{X}\}, \tag{Π}$$

with $\nu(x,y) = (\nu_1(x,y), \ldots, \nu_q(x,y))$, where $\nu_i(x,y) = c^i x + h^i y, i \in \{1, \ldots, q\}$, and $\mathcal{X} = \{(x,y) \in \mathbb{Z}^n \times \mathbb{R}^p \mid Ax + Gy \leq b, x \geq 0, y \geq 0\}$. Let $\mathcal{X}_E$ be the efficient set.

The parameterized MO-MILP replaces the objective function coefficients c^i by $c^i + \Delta^i, i \in \{1, \ldots, q\}$

$$\max\{\nu_\Delta(x,y) \mid (x,y) \in \mathcal{X}\}, \tag{Π_Δ}$$

with $\Delta = (\Delta^i : i \in \{1, \ldots, q\}) \in \mathbb{R}^{nq}$ and $\nu_{\Delta,i}(x,y) = c^i x + h^i y + \Delta^i x^i$, $i \in \{1, \ldots, q\}$. Let $\mathcal{X}_E^\Delta$ be the corresponding efficient set.

Definition 3: The sensitivity region

The sensitivity region $\Omega(\mathcal{X})$ is the region of changes $\Delta \in \mathbb{R}^{nq}$ to the coefficients such that

- efficient solutions retain the component-wise ordering, and
- inefficient solutions in $\mathcal{X}$ remain inefficient.

Note that $\Omega(\mathcal{X}) \subseteq \{\Delta \in \mathbb{R}^{nq} \mid \mathcal{X}_E = \mathcal{X}_E^\Delta\}$

Changes to a single coefficient

Consider the $(q+1)$-objective auxillary problems

$$
\begin{aligned}
\max\ & (\operatorname{sgn}(c_k^i)x_k, \sum_{j\neq k} c_j^i x_j + h^i y, \\
& c^1x + h^1y, \dots, c^{i-1}x + h^{i-1}y, c^{i+1}x + h^{i+1}y, \dots, c^qx + h^qy) \\
\text{st}\ & (x,y) \in \mathcal{X} \\
\max\ & (-\operatorname{sgn}(c_k^i)x_k, \sum_{j\neq k} c_j^i x_j + h^i y, \\
& c^1x + h^1y, \dots, c^{i-1}x + h^{i-1}y, c^{i+1}x + h^{i+1}y, \dots, c^qx + h^qy) \\
\text{st}\ & (x,y) \in \mathcal{X}
\end{aligned}
$$

and denote by $\mathcal{X}_E^+$ and $\mathcal{X}_E^-$ the efficient sets, respectively.

Theorem 3

For $c_k^i \neq 0$, $\Omega(\mathcal{X}) = \Omega(\mathcal{X}_E^+) \cup \Omega(\mathcal{X}_E^-)$.

Changes to K coefficients requires at most 2^K $(q+K)$-objective auxillary problems.

Capital budgeting

Invest in projects by optimally allocating limited funds to suggested projects.

The problem can be formulated as a bi-objective ILP:

$$\begin{aligned}
&\max\left\{\left(\mathbb{E}\left[\sum_{j=1}^{n} V_j x_j\right], -\mathbb{V}\left[\sum_{j=1}^{n} V_j x_j\right]\right) \middle| \sum_{j=1}^{n} a_j x_j \leq b, x_j \in \{0,1\}, j = 1, \ldots, n\right\} \\
&= \max\left\{\left(\sum_{j=1}^{n} \mu_j x_j, -\sum_{j=1}^{n} \sigma_j^2 x_j\right) \middle| \sum_{j=1}^{n} a_j x_j \leq b, x_j \in \{0,1\}, j = 1, \ldots, n\right\},
\end{aligned}$$

where $x_j = 1$ if project j is selected and $x_j = 0$ otherwise. Moreover, $\mu_j = \mathbb{E}[V_j]$ and $\sigma_j^2 = \mathbb{V}[V_j]$ are the expected value and variance of NPV of future cashflows, respectively, a_j is the initial cashflow, and b is the initial budget.

Capital budgeting

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
μ_j	165	123	123	176	81	145	156	119	86	63	142	93	125	101	31	57	47	7
σ_j^2	70	123	124	182	99	217	235	216	166	135	371	251	355	325	160	306	415	418
a_j	125	77	111	121	128	77	103	109	124	118	78	108	124	134	149	127	116	132
b	1086																	

Table: Rosenblatt and Sinuany-Stern (1989)

- The sensitivity interval for μ_1 is $(-165, \infty)$. With low risk, project 1 belongs to all efficient portfolios with positive expected value.
- The sensitivity interval for μ_7 is $(-11, 5)$. Project 6 becomes superior to project 7 if the expected value becomes less than or equal to 145, with the same expected value but lower risk.
- For simultaneous changes to two coefficients, the sensitivity region may no longer be convex.

Bi and tri-objective binary and integer knapsack problems

Instance				CPU				Cardinality				
obj	class	changes	n	initial	sensitivity	total	std	$\lvert\mathcal{X}_E\rvert$	$\lvert\bar{\mathcal{X}}\rvert$	aux	red	LB($\mathcal{X}$)
2	bin	1	100	5.2	15.7	20.9	10.5	107	76	1.2	0	$5.0 \cdot 10^{25}$
			300	77.4	351.1	428.5	88.3	713	627	1.5	2	$4.4 \cdot 10^{79}$
			400	150.4	485.1	635.4	291.5	1174	527	1.0	0	$1.4 \cdot 10^{105}$
			500	159.2	2148.1	2307.3	1858.8	1589	1098	1.2	24	$7.5 \cdot 10^{131}$
		2	100	4.9	23.2	28.0	14.7	107	159	1.2	13	$5.0 \cdot 10^{25}$
			300	54.4	610.1	664.6	279.1	713	1641	1.0	0	$4.4 \cdot 10^{79}$
			400	117.2	1058.7	1175.9	496.0	1174	1957	1.0	0	$1.4 \cdot 10^{105}$
2	int	1	100	25.5	269.5	295.0	206.6	339	1 417	1.7	0	
			300	103.7	1 891.4	1 995.1	836.7	1 559	8 613	1.5	93	
		2	100	11.5	1 182.3	1 193.8	537.8	243	6 748	2.4	1338	
3	bin	3	40	20.4	91.1	111.4	54.2	277	519	1.2	51	
			80	205.9	3 234.6	3 439.7	3 632.4	2 236	3 551	1.5	361	
3	int	3	40	49.1	9 770.0	9 819.1	6 695.4	750	15 337	3.8	5617	
			50	225.1	13 653.5	13 878.5	6 561.2	2 938	17 478	4.2	10 258	

The end

Thank you for the attention!